ON THE BOREL-CANTELLI LEMMA

BY

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ABSTRACT

A new variant of the "divergent" part of the Borel-Cantelli lemma for events derived from a Markov chain is given. Further two applications are considered. One of the applications refers to the denumerable Markov chain and the second is a new proof of the "strong" theorem corresponding to the "arc sine law".

1. Introduction

The celebrated Borel-Cantelli lemma ([2] and [4]) asserts that if $A_1, A_2, \cdots$ is a sequence of arbitrary events and if $\sum_{n=1}^{\infty} P(A_n) < \infty$ then the probability of the event

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

is 0. If, in addition the events $A_1, A_2, \cdots$, are assumed to be mutually independent and if $\sum_{n=1}^{\infty} P(A_n) = \infty$ then $P(\limsup A_n) = 1$.

Although this lemma has a large area of applicability, in many interesting cases the assumption of independence fails to hold, such that extensions to dependent events of the second part of the Borel-Cantelli lemma prove useful. In this respect a notable example is provided by the events derived from the partial sums of independent random variables. It has been often noticed that the sequence of the partial sums of independent random variables forms a Markov chain with complicated transition probabilities.

Several authors, among which E. Borel [3], P. Levy [9] p. 249, M. Loève [11], K. Chung and P. Erdös [5], S. Nash [12], J. Blum, D. Hanson and L. Koopmans [1] and M. Iosifescu [8] gave extensions of the Borel-Cantelli lemma to some dependent events.

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Nevertheless, there are still many cases when these extensions are either not workable or lead to complicated proofs.

The aim of this paper is to give a variant of Borel-Cantelli lemma that seems to have a wide applicability to Markov chains and in particular to the partial sums of independent random variables. Subsequently, we shall give an application to denumerable Markov chains and a relatively simple proof of a known result as to the partial sums of independent random variables.

THEOREM 1. *Let $\{X_n;\, n \geqq 1\}$ be a real valued Markov chain, $\mathscr{F}_m^n$ the σ-algebra generated by the random variables $X_m, \cdots, X_n$ $(m, n = 1, 2, \cdots)$. Let further $\{A_n : n \geqq 1\}$ be a sequence of events with the property that for any n $(n = 1, 2, \cdots)$ there exists a number $r(n)$ such that $A_n \in \mathscr{F}_1^{r(n)}$ and either $A_{n+1}, A_{n+2}, \cdots \in \mathscr{F}_{r(n)+1}^{\infty}$ or there exists a number $s(n)$ with $s(n) > n$ such that $A_n \cap A_i = \varnothing$ for $n < i \leqq s(n)$ and $A_{s(n)+1}, A_{s(n)+2}, \cdots, \in \mathscr{F}_{r(n)+1}^{\infty}$, $\{r(n) : n \geqq 1\}$ being an increasing sequence of positive integers. Denote $\mathscr{F}_n^n = \mathscr{F}_n$ and*

$$a_l = \limsup_{n \to \infty} \left[\sup_{A \in \mathscr{F}_{r(n+l)}} (P\{A\} - P\{A \mid A_n\}) \right] \tag{1}$$

Then $\{a_l : l \geqq 1\}$ is a decreasing sequence. If, in addition.

(I) $\sum_{n=1}^{\infty} P(A_n) = \infty$

(II) *There exists a number δ, $0 \leqq \delta < 1$ such that*

$$\lim_{l \to \infty} a_l \leqq \delta$$

Then $P\{\limsup A_n\} \geqq 1 - \delta$.

PROOF. We shall prove firstly that the sequence $\{a_l : l \geqq 1\}$ is monotone. For that, let us consider an arbitrary N-dimensional Borelian set B and write $r(n + l) = r'$.

Denote

$$A = \{(X_{r'+2}, \cdots, X_{r'+N+1}) \in B\}$$

$$P^{r'+1}\{A\} = P\{X_{r'+1} \in A\}$$

$$P^{r'+1}\{x;\, A_n\} = P\{X_{r'+1} = x \mid A_n\}$$

We get

$$(2) \qquad P\{A\} = \int P\{A \mid X_{r'+1} = x\} P^{r'+1}\{dx\}$$

$$(3) \qquad P\{A \mid A_n\} = \int P\{A \mid X_{r'+1} = x\} P^{r'+1}\{dx; A_n\}$$

the last relationship being a consequence of the Chapman-Kolmogorov formula.

Subtracting (3) from (2) and applying a standard approximation reasoning we obtain

$$\sup_{A \in \mathscr{F}^{\infty}_{r(n+l)+2}} (P\{A\} - P\{A \mid A_n\}) \leqq \sup_{A \in \mathscr{F}r_{(n+l)+1}} (P\{A\} - P\{A \mid A_n\})$$

Taking now into account that $\{r(n): n \geqq 1\}$ is an increasing sequence, we conclude the first part of the theorem.

Let us notice further that under the assumption (II) for any $\varepsilon > 0$ we may find a number m sufficiently large such that $a_m \leqq \delta + \varepsilon/2$.

We split now the initial sequence $\{A_n: n \geqq 1\}$ into the following subsequences

$$A_1, A_{m+1}, \cdots$$

$$A_2, A_{m+2}, \cdots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_m, A_{2m}, \cdots$$

From (I) follows that in the above array there exists at least one row such that $\sum_{k=1}^{\infty} P(A_{mk+i}) = \infty$

Putting $A_{nm+i} = A_n'(n = 1, 2, \cdots)$ and noticing that in the case when $A_k' \cap A_j' = \varnothing$ for $k < j \leqq s(k)$, $\bar{A}_j'$ can be omitted in what follows, we get for n sufficiently large

$$\begin{aligned} P\left(\bigcup_{i=n}^{n+N} A_i'\right) &= P(A'_{n+N}) + P(\bar{A}'_{n+N} A'_{n+N-1}) + \cdots + P(\bar{A}'_{n+N} \cdots \bar{A}'_{n+1} A'_n) \\ &\geqq P(A'_{n+N}) + P(\bar{A}'_{n+N})(A'_{n+N-1}) - (\delta + \varepsilon) P(A'_{N+n-1}) + \cdots \\ &\quad + P(A'_n) P(\bar{A}'_{n+N} \cdots \bar{A}'_{n+1}) - (\delta + \varepsilon) P(A'_n) \\ &= \sum_{i=n}^{n+N} P(A_i') [P(\bar{A}'_{n+N} \cdots \bar{A}'_{n+1}) - (\delta + \varepsilon)] \end{aligned}$$

We deduce easily that

$$\lim_{n \to \infty} P\left(\bigcap_{i=n}^{\infty} \bar{A}_i\right) \leqq \delta$$

wherefrom we get

$$P(\limsup A_n) = \lim_{n\to\infty} P\left(\bigcup_{i=n}^{\infty} A_i\right) \geqq 1-\delta$$

which concludes the proof

2. Application to denumerable Markov chains

THEOREM 2. *Let $\{X_n: n \geqq 1\}$ be a denumerable inhomogeneous Markov chain, $\{a_i: i \geqq 1\}$ the set of its states, $p_i^{(n)} = P\{X_n = a_i\}$, $P_{ij}^{(n,k)} = P\{X_{n+k} = a_j \mid X_n = a_i\}$ and B a subset of states .If $\sum_{n=1}^{\infty} \sum_{(j:a_j \in B)} p_j^{(n)} = \infty$, then*

$$P\{X_n \in B \text{ infinitely often}\} \geqq 1-\delta$$

where

$$\delta = \lim_{l\to\infty} \limsup_{n\to\infty} \frac{\sum_{i=1}^{\infty} \sum_{(j:a_j \in B)} p_j^{(n)}(p_i^{(n+l)} - p_{j,i}^{(n,l)})^+}{\sum_{(j:a_j \in B)} p_j^{(n)}}$$

The proof follows directly from Theorem 1.

3. Application to the arc sine law

We now give a simple proof of the "divergent" part of the "strong theorem" corresponding to the arc sine law. This theorem was proved using a Borel-Cantelli type lemma by Chung and Erdös [5].

THEOREM 3. *Let $\{X_n: n \geqq 1\}$ be a sequence of independent random variables and each X_n assume the values $+1$ and -1 with probabilities* $\frac{1}{2}$ and $\frac{1}{2}$. *Let $S_n = \sum_{i=1}^{n} X_i$ and N_n the number of positive sums among $S_1, S_2, \cdots, S_n$. If $\psi(n)$ is an increasing function of n and if*

$$\sum_{n=1}^{\infty} \frac{1}{n\psi(n)^{\frac{1}{2}}} = \infty$$

then

$$P\left\{N_n \leqq \frac{n}{\psi(n)} \text{ i.o.}\right\} = 1 \tag{5}$$

PROOF. As in [5] we shall consider the events

$$E_k = \{S_{2k} = 0,\ S_i < 0 \quad \text{for } 2k < i \leqq 2k\psi(k)\} \qquad (k = 1, 2, \cdots) \tag{6}$$

Then $P\{E_k\} \sim Ak^{-\frac{1}{2}}$, A being a positive constant, and to prove (5) it will be sufficient to show that $P\{E_k \text{ i.o.}\} = 1$ ([5]).

From (6) we get that, for $m \leqq k\psi(k)$ and $m > k$, $P\{E_k E_m\} = 0$.

Therefore, to prove the theorem by applying Theorem 1 to the sequence of events $\{E_k : k \geqq 1\}$ derived from the Markov chain $\{S_n : n \geqq 1\}$, it will be enough to show that the condition (II) is satisfied.

Considering (1) for $l = 1$, we are led to the expression

$$\sum_{i=-(2k\psi(k)+1)}^{0} (P\{S_{2k\psi(k)+1} = i\} - P\{S_{2k\psi(k)+1} = i \mid S_{2k} = 0,\ S_i < 0,\ 2k < i \leqq 2k\psi(k)\})^{+} \tag{7}$$

The condition (II) will be satisfied if we shall prove that for any a and b with $a < 0$, $b < 0$, $a < b$ and k sufficiently large, we have

$$\sum_{i \in A_k} (P\{S_{2k\psi(k)+1} = i\} - P\{S_{2k\psi(k)+1} = i \mid S_{2k} = 0, S_i < 0, 2k < i \leqq 2k\psi(k)\})^{+} \leqq \alpha \tag{8}$$

with $\alpha < \frac{1}{2}$, where $A_k = (a\sqrt{2k\psi(k)+1},\ b\sqrt{2k\psi(k)+1})$.
But it is easy to see that

$$\begin{aligned} P\{S_{2k\psi(k)+1} &= i \mid S_{2k} = 0,\ S_i < 0,\ 2k < i \leqq 2k\psi(k)\} \\ &= P\{S_{2k(\psi(k)-1)+1} = i,\ S_1 < 0, \cdots, S_{2k(\psi(k)-1)} < 0\}\ \frac{P\{S_{2k} = 0\}}{P\{E_k\}} \end{aligned} \tag{9}$$

On the other hand

$$\begin{aligned} &P\{S_{2k(\psi(k)-1)+1} = i,\ S_1 < 0, \cdots, S_{2k(\psi(k)-1)} < 0\} \\ &= \frac{|i|}{2k(\psi(k)-1)} \binom{2k(\psi(k)-1)}{\frac{2k(\psi(k)-1)+i}{2}} 2^{-2k(\psi(k)-1)} \end{aligned} \tag{10}$$

the last equality being a consequence of theorem 1, p. 70 [6].

As for the first term of (8), one has

$$P\{S_{2k\psi(k)+1} = i\} = \binom{2k\psi(k)+1}{\frac{2k\psi(k)+1+i}{2}} 2^{-(2k\psi(k)+1)} \tag{11}$$

Reminding now that $P\{E_k\} \sim Ak^{-1}(\psi(k))^{-\frac{1}{2}}$, $P\{S_{2k} = 0\} \sim (1/\pi k)^{\frac{1}{2}}$, and making use of (9) and (10), we get for k sufficiently large

$$\begin{aligned} &P\{S_{2k\psi(k)+1} = i \mid S_{2k} = 0,\ S_i < 0,\ 2k < i \leqq 2k\psi(k)\} \\ &\sim \frac{C|i|}{k^{\frac{1}{2}}\psi^{\frac{1}{2}}(k)} \binom{2k(\psi(k)-1)}{\frac{2k(\psi(k)-1)+i}{2}} 2^{-2k(\psi(k)-1)} \end{aligned} \tag{12}$$

C being a positive constant. Using now the well known binomial approximation to the normal distribution, we get (8). Therefore, $\delta < 1$. Employing now the $0 - 1$ law, we conclude the proof.

REMARK. 1) It is possible to use Theorem 1 for proving the above Theorem under suitable conditions in the case of random variables having continuous distribution functions.

2) It would be interesting to find the conditions under which Theorem 1 applies to derive "strong bounds" for sequences of independent random variables of the type considered in [8]. Using the Chung-Erdös variant of Borel-Cantelli lemma, such a way was follows by M. Lipschutz [10].

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